Spacecraft initial conditions are

$$R_0 = 1000\,\text{km}, \qquad V_{r0} = -5\,\text{km/s}, \qquad V_{\theta 0} = 25\,\text{km/s}$$

and the spacecraft guidance gains are

$$c_1 = 0.01, \qquad c_2 = 0.06, \qquad c_3 = 0.0009$$

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Stabilization via Dynamic Output Feedback: A Numerical Approach

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I. Introduction

IN the design of controllers, the complete state is often not available for feedback. One can reconstruct the state using a Luenberger observer,[1] but this may unduly complicate the structure of the controller due to the large dimension of the observer dynamics. (If n is the dimension of the state and p the number of measured states, then the observer will be of order n-p.) In Ref. 2, a method is presented for determining constant output feedback gains for the control of systems with inaccessible states. The control is linearly dependent on the measured outputs. However, it may not be possible to achieve stabilization with constant output feedback; a spring-mass system with position as output is the classic example.

Here, we present a numerical method for determining a dynamic output feedback controller where the controller order is less than the (n-p)th order controller required in Ref. 1. Such a controller has the capability of stabilizing systems where static output feedback fails while being simpler in structure than the control based on the separation theorem, where one first designs the controller gains assuming the complete state can be measured and then uses the same gains with an estimated state. The procedure is extended to the problem of simultaneously stabilizing several plants by a single controller.

II. Problem Formulation

Consider a linear, time-invariant system described by

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad y(t) = Cx(t) \tag{1}$$

where $x(t) \in R^n$, $u(t) \in R^m$, and $y(t) \in R^p$ are the state, control, and measured output, respectively. It is assumed that the system is controllable and observable.

Static output feedback problem: Determine an $m \times p$ matrix K such that

$$u(t) = Ky(t) \tag{2}$$

stabilizes Eqs. (1); i.e., the eigenvalues of $A + BKC$ lie in the left half of the complex plane.

Dynamic output feedback problem: Let $z(t) \in R^q$ satisfy

$$\dot{z}(t) = Fz(t) + Gy(t) \tag{3}$$

Determine the dimension q and the matrices $K_1 \in R^{m\times q}$ and $K_2 \in R^{m\times p}$ such that

$$u(t) = K_1 z(t) + K_2 y(t) \tag{4}$$

stabilizes Eqs. (1) and (3).

For the static output feedback problem, it is shown in Ref. 3 that max $\{m,p\}$ eigenvalues can be assigned. Thus, in general, stability cannot always be achieved via static output feedback because nothing can be done about the remaining $n - \max\{m,p\}$ eigenvalues. However, it may still be possible to stabilize the system using output feedback, but the possibility of accomplishing this must be ascertained on a problem by problem basis; no general theory is available.

For the dynamic output feedback problem, one question that arises is the choice of the smallest dimension q of the dynamic compensator needed to achieve stability. There is no theoretical result specifying the minimum order of the compensator required for pole assignability (or even stabilizability). In Ref. 4, it is shown that this order is at most equal to $\min\{\rho_c, \rho_o\}$, where ρ_c and ρ_o are the controllability and observability indices, respectively, defined by

$$\rho_c = \min\{\rho: \ \text{rank}[B \quad AB \quad \dots \quad A^\rho B] = n\}$$

$$\rho_o = \min\{\rho: \ \text{rank}[C' \quad A'C' \quad \dots \quad (A')^\rho C'] = n\}$$

Kimura[5] states that if the order q of the dynamic compensator is chosen to be $q = n - m - p + 1$ then arbitrary eigenvalue assignment is possible. Both of these results are sufficient conditions for arbitrary eigenvalue assignment; it may be possible to achieve eigenvalue assignment or stability by a compensator of lower order than required by either theory.

In the next section, we review the numerical technique of Ref. 2 for solving the static output feedback problem and give some modifications of the technique that improves convergence. This procedure involves the minimization of a suitable objective function. Then, we show that the dynamic output

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feedback problem can be embedded in a static output feedback problem of higher dimension. Thus, the algorithm for solving the static output feedback problem can also be used for solving the dynamic output feedback problem. In Sec. IV, we apply the algorithm to several aerospace examples.

The extension of the technique to the problem of designing a single controller to stabilize a system at several operating conditions—the simultaneous stabilization problem—is discussed and illustrated via examples in Sec. V.

III. Minimization Problem

Consider first the static output feedback problem of determining K in Eq. (2) to stabilize Eqs. (1). We base our approach on Ref. 2 where the main idea is to minimize the distance between the set of actual eigenvalues of $A + BKC$ and the set of desired eigenvalues, denoted by $\{\mu_1, \mu_2, \ldots, \mu_n\}$. Since the eigenvalues of $A + BKC$ depend on the choice of K, we will denote these eigenvalues by $\{\lambda_1(K), \lambda_2(K), \ldots, \lambda_n(K)\}$. In order to measure this distance, an ordering is imposed on the sets of eigenvalues. This is done by sweeping the real axis from $-\infty$ and increasing the subscripts as we proceed to $+\infty$. Thus, $\mathrm{Re}(\mu_1) \leq \mathrm{Re}(\mu_2) \leq \cdots \leq \mathrm{Re}(\mu_n)$. If two or more eigenvalues have the same real part, they are labeled according to the value of their imaginary parts, the one with the largest imaginary part having the lowest index. The eigenvalues $\lambda_i(K)$ are ordered similarly.

For a given K, let

$$d_i^2 \triangleq |\lambda_i(K) - \mu_i|^2 = \left\{\mathrm{Re}[\lambda_i(K)] - \mathrm{Re}(\mu_i)\right\}^2 + \left\{\mathrm{Im}[\lambda_i(K)] - \mathrm{Im}(\mu_i)\right\}^2$$

In Ref. 2, K is chosen to minimize

$$\tilde{J}(K) = \sum_{i=1}^{n} d_i^2 \tag{5}$$

Here, we modify the cost by introducing relative distance r_i, $r_i \triangleq d_i / |\mu_i|$, and a positive penalty constant c_i. Thus, our performance index is

$$J(K) = \sum_{i=1}^{n} c_i r_i^2 \tag{6}$$

This modification was suggested in Ref. 6 for a different problem and led to better numerical convergence than that achieved with the cost function in Eq. (5). The default value of c_i is 1. Since we want the eigenvalues to be stable, if $\lambda_i(K)$ is unstable (real part greater than or equal to zero), we set $c_i = 1000$. This tends to force the eigenvalues into the left half plane. A stability margin σ is enforced by setting $c_i = 100$ if eigenvalue i has real part greater than $-\sigma$. It is also possible to enforce a damping ratio by an appropriate weighting c_i, and we allow for that option in our algorithm.

To solve the problem of minimizing $J(K)$, the polytope pattern search method (or flexible polyhedron method) is used.[7] A positive feature of this algorithm is that it does not require a gradient calculation. However, we found that the algorithm is quite inefficient (slow to converge) and does not guarantee convergence to the absolute minimum. This problem of false convergence was frequently encountered and could sometimes be solved by choosing different starting points.

As an alternative to the flexible polyhedron method, we tried the Levenberg-Marquardt method.[8,9] When it converged, this algorithm converged much faster than the flexible polyhedron method (e.g., 9 iterations vs 1017 iterations). However, it has several drawbacks. It requires the computation of a Jacobian matrix, which in turn requires $c_i = 1$ for all i. Also, it does not converge if repeated eigenvalues of $A + BK$ occur at any step of the iteration. Finally, if there is a change in the ordering of the eigenvalues in going from step i to step $i+1$, the method fails to converge. Both of these convergence failures are due to the cost function becoming discontinuous. The flexible polyhedron method is not sensitive to discontinuities in the cost function. Therefore, we used a combination of the flexible polyhedron method and the Levenberg-Marquardt method. If the Levenberg-Marquardt method encounters a critical point, we switch to the flexible polyhedron method for a few steps and then switch back to the Levenberg-Marquardt method. Often only a few steps of the flexible polyhedron method are needed to overcome a critical point. A program using this algorithm has been implemented for a PC.[10]

For systems that cannot be stabilized by static output feedback, it is necessary to introduce dynamics into the control. Consider again systems (1) and (3), with control given by Eq. (4), $u(t) = K_1 z(t) + K_2 y(t)$. We create a new problem

$$\dot{\tilde{x}}(t) = \begin{bmatrix} A & O \\ O & O \end{bmatrix} \tilde{x}(t) + \begin{bmatrix} B & O \\ O & I \end{bmatrix} \tilde{u}(t) \tag{7a}$$

$$\tilde{y}(t) = \begin{bmatrix} C & O \\ O & I \end{bmatrix} \tilde{x}(t) \tag{7b}$$

where $\tilde{x}(t) \in R^{n+q}$, $\tilde{u}(t) \in R^{m+q}$, and $\tilde{y}(t) \in R^{p+q}$.

It has been shown in Ref. 4 that, if

$$\tilde{u}(t) = K\tilde{y}(t) \tag{8}$$

is a static output feedback stabilizing control for Eqs. (7), then, with K partitioned as

$$K = \begin{bmatrix} K_2 & K_1 \\ G & F \end{bmatrix} \tag{9}$$

the matrices G, F, K_2, K_1 define a dynamic feedback stabilizing control for Eqs. (1) via Eqs. (3) and (4).

Thus, the dynamic compensator problem can be solved by solving a related static output feedback problem, and the numerical method developed for static output feedback problems can be used. Our approach is to choose the order of the compensator q and numerically check if stabilization can be achieved. If not, increase the order of the compensator by 1 and repeat the calculations. An upper bound on q is given by the minimum of the numbers, $(n - p - m + 1)$, ρ_c, and ρ_o.

In the next section, we illustrate our approach with two examples.

IV. Examples

In the first example, we show that exact eigenvalue placement can be achieved with a dynamic compensator of order $n - m - p + 1$. The second example demonstrates that stability (but not exact eigenvalue assignment) can be achieved with a compensator of order less than $\min\{\rho_c, \rho_o, n - p - m + 1\}$.

Example 1

The pitch axis model of the AFT1/F-16 flying at 3000 ft and Mach number 0.6 is considered.[11] The system model is

$$\dot{x}(t) = \begin{bmatrix} 0 & 1.00 & 0 \\ 0 & -0.87 & 43.22 \\ 0 & 0.99 & -1.34 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ -17.25 & -1.58 \\ -0.17 & -0.25 \end{bmatrix} u(t)$$

where x_1 is the pitch angle, x_2 the pitch rate, and x_3 the angle of attack. The control vector components are elevator deflection and flaperon deflection and the output is pitch angle,

$y = [1 \quad 0 \quad 0]x$. If a low-order observer is used, its dimension will be $n - p = 3 - 1 = 2$. With the theory from Ref. 5, exact pole assignment is possible with a compensator of order $n - m - p + 1 = 1$. We use our numerical algorithm to accomplish this. From Ref. 11, the desired locations for the eigenvalues are $-5.6 \pm 4.2j, -1$ and we place the compensator eigenvalue at -10. Solving numerically, we obtain the first-order compensator

$$\dot{z}(t) = -19.99z(t) + 3.48y(t)$$

$$u(t) = \begin{bmatrix} -29.46 \\ -159.79 \end{bmatrix} z(t) + \begin{bmatrix} 8.10 \\ 22.03 \end{bmatrix} y(t)$$

It is possible to stabilize the system with static output control, but exact eigenvalue assignment cannot be achieved. For static output feedback, our algorithm for minimizing the weighted distance between the actual eigenvalues and the desired eigenvalues led to $u^T(t) = [4.38 \quad -9.29]y(t)$ with eigenvalues at $\{-0.737 \pm 4.2, -0.737\}$.

Example 2

For our second example, we use the track guided bus model of Ref. 12. The five state variables $x = [\alpha, \epsilon, \epsilon, y, \beta]^T$ are defined in Fig. 1. The linearized dynamics for small deviations of the bus from the guide line are

$$\dot{x}(t) = \begin{bmatrix} -33.4 & -0.986 & 0 & 0 & 0.309 \\ 0.525 & -0.639 & 0 & 0 & 2.103 \\ 0 & 1 & 0 & 0 & 0 \\ 20 & 6.12 & 20 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

and the output is $y(t) = [0 \quad 0 \quad 0 \quad 1 \quad 0]x(t)$. The theory from Refs. 4 or 5 states that with a compensator of order 4, arbitrary eigenvalue assignment is possible. Instead, we try to stabilize the system with a second-order compensator. Numerical calculations led to

$$\begin{bmatrix} \dot{z}_1(t) \\ \dot{z}_2(t) \end{bmatrix} = \begin{bmatrix} -0.935 & -2.419 \\ 0.843 & 1.210 \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} + \begin{bmatrix} -1.861 \\ -4.058 \end{bmatrix} y(t)$$

$$u(t) = -0.2468z_1(t) - 0.2901z_2(t) + 1.515y(t)$$

and this controller stabilizes the system. Thus, if stability is the only requirement, it can be attained with a second-order compensator.

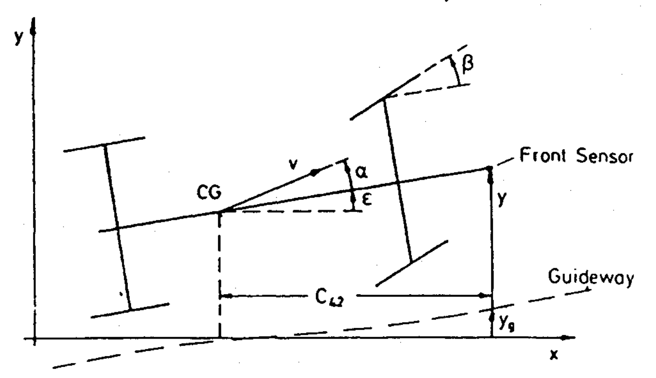

Fig. 1 Definition of variables for the track guided bus.

V. Simultaneous Stabilization

Here, we consider the problem of determining a single compensator that stabilizes several plants. This problem can occur when the true system deviates from the assumed model due to changing parameters or because of inexact modeling. This single-controller concept can also be used to design one controller for a system operating at several different conditions rather then having to use several controllers and a gain scheduling procedure.

Consider P plants,

$$\dot{x}(t) = A_i x(t) + B_i u(t), i = 1,2,\ldots,P$$

The problem is to determine a compensator order q and matrices F, G, K_1, and K_2 such that

$$\begin{bmatrix} \dot{x}(t) \\ \dot{z}(t) \end{bmatrix} = \begin{bmatrix} A_i B K_2 C & B_i K_1 \\ GC & F \end{bmatrix} \begin{bmatrix} x(t) \\ z(t) \end{bmatrix} \tag{10}$$

is asymptotically stable for $i = 1,2,\ldots,P$; i.e.,

$$u(t) = K_1 z(t) + K_2 y(t), \qquad \dot{z}(t) = Fz(t) + Gy(t)$$

stabilizes all P plants.

The technique of Sec. III can be used for the simultaneous stabilization problem by minimizing the sum of the objective functions for each system:

$$J(K) = \sum_{i=1}^{P} J_i(K) \tag{11}$$

Of course, exact eigenvalue placement for each plant is almost never possible. Nevertheless, by trying to make the eigenvalues of each plant close to a desired set, we can sometimes achieve stability. We illustrate via an example.

In Ref. 13, a model for F4-E aircraft at different flight conditions is given. The system is described as

$$\dot{x}(t) = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & -14 \end{bmatrix} x(t) + \begin{bmatrix} b_1 \\ 0 \\ 14 \end{bmatrix} u(t)$$

$$y(t) = [1 \quad 12.43 \quad 0]x(t)$$

The state vector $x = [n_z, q, \delta_e]^T$, where n_z is the normal acceleration, q the pitch rate, and δ_e the deviation of the elevator deflection.

Table 1 Data for F4-E Aircraft

Parameter	Flight condition 1: Mach = 0.5, Altitude = 5000 ft	Flight condition 2: Mach = 0.85, Altitude = 5000 ft	Flight condition 3: Mach = 0.9, Altitude = 35,000 ft	Flight condition 4: Mach = 1.5, Altitude = 35,000 ft
a_{11}	−0.9896	−1.702	−0.667	−0.5162
a_{12}	17.41	50.72	18.11	26.96
a_{13}	96.15	263.5	84.34	178.9
a_{21}	0.2648	0.2201	0.08201	−0.6869
a_{22}	−0.08512	−0.6587	−0.6587	−1.225
a_{23}	−11.39	−31.99	−10.81	−30.38
b_1	−97.78	−272.2	−85.09	−175.6

There are four flight conditions selected out of the possible range, whose parameters are given in Table 1.

According to Refs. 4 and 5, a dynamic compensator of order 2 guarantees exact pole assignment for one flight condition. Therefore, we try to stabilize all four flight conditions with a second-order dynamic compensator.

Numerically, we compute the matrix K of Eq. (8) that minimizes Eq. (11) for the augmented system (10) and obtain

$$K = \begin{bmatrix} 0.0288 & 0.0191 & 0.0127 \\ -3.7633 & -4.2102 & 3.0319 \\ 4.7533 & -0.4610 & -3.5571 \end{bmatrix}$$

From Eq. (9), the corresponding control is

$$u(t) = (0.0191 \;\; 0.0127)z(t) + 0.0288y(t)$$

where

$$\dot{z}(t) = \begin{bmatrix} -4.2102 & 3.0319 \\ -0.4610 & -3.5571 \end{bmatrix} z(t) + \begin{bmatrix} -3.7633 \\ 4.7533 \end{bmatrix} y(t)$$

This control stabilizes all four flight conditions.

Although the proposed numerical procedure for solving the simultaneous stabilization problem will not always lead to a solution, this example illustrates that it does offer a possible method and can be included in our toolbox of techniques.

VI. Conclusions

A numerical procedure for determining a dynamic output feedback controller was presented. Such a controller has the advantage of being of lower order than a controller based on the separation theorem where an estimate of the entire state is used. It can also produce a stable system when static output feedback fails. The procedure was extended to the problem of the simultaneous stabilization of several plants by a single controller.

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Algebraic Approach to the Bearings-Only Estimation Equations

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Introduction

THE problem of interest is that of locating a target from line-of-sight bearing measurements performed from a moving platform. Most recursive algorithms implement some variant of the extended Kalman filter (EKF).[1-3] Common to the EKFs thus far described in the literature is that the measurement equation yields a bearing angle relative to some known axis. The filter equations, as a consequence, are expressed in terms of the transcendental arctangent function. Though mathematically correct, the need to calculate the arctangent function imposes demands on the sample period of a real-time estimation system, especially if a numerical integration or other type of iteration is required. Furthermore, the extension of the equations to the three-dimensional problem is complicated, and consideration of more than one measurement array axis is cumbersome.

In this Note, two examples of filter equations are derived for which the measurement equation yields the directional cosine of the target relative to a known axis. (The directional cosine measurement is the natural output of many angle sensors and is often unnecessarily converted to angle by applying the arccosine function.) This redefinition of the measurement equation leads to filter equations that are nonlinear algebraic rather than transcendental. These equations are, furthermore, inherently three-dimensional. Inclusion of multiple measurement axes is straightforward.

Algebraic Filter Equations

To demonstrate the utility of these algebraic derivations to bearings-only estimation, two EKFs are presented. The two EKFs differ in their state variable assignment. The first filter to be derived has a linear measurement equation and a nonlinear state propagation equation. The second filter has a nonlinear measurement equation and linear state propagation equation. The following symbols are used in the derivations: $\boldsymbol{R}$ is the unknown line of sight vector from the platform to the target; $|\boldsymbol{R}|$ the unknown range from the platform to the target; $\boldsymbol{r}$ the unknown line-of-sight unit vector, $= \boldsymbol{R}/|\boldsymbol{R}|$; $\boldsymbol{V}$ the known velocity vector of the platform; and $\boldsymbol{a}$ the known unit vector along the measurement axis.

Case 1: Nonlinear-in-State, Linear Measurement Extended Kalman Filter

Noting that the three components of $\boldsymbol{r}$ are not independent, the (pseudo) state vector is defined as

$$\boldsymbol{x} = \begin{bmatrix} |\boldsymbol{R}| \\ \boldsymbol{r} \end{bmatrix}$$

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